Intrinsic Description of Ring Spinning Balloon mechanics

by

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Abstract

A brief description of the intrinsic mechanical properties of dynamical space curves is provided and it is applied to the spinning balloon, which in general represents a space curve with a time-dependent shape. In particular, the intrinsic frame-dependent forces experienced by mass elements of fibers (as an element of the spinning balloon) in the local natural comoving frame of reference are characterized. The preliminary results obtained suggest that it may be possible to develop new practical diagnostic and forecasting tools for the ring spinning process.

1. Introduction

As pointed out in a paper by Gosh, Batra, Dua, and Zeidman at the 1991 meeting of this conference, attempts to understand ring spinning theoretically date back to 1881. In that paper, the more recent papers by Batra, Ghosh, and Zeidman were discussed which reformulated this problem in the late 1980's. Basically, they divided the dynamical problem of the spinning process into four regions: (Ⅰ) from the front roller to the pig-tail, (Ⅱ) from the pig-tail to the traveler (in the absence of balloon control rings), (Ⅲ) through the traveler, and (Ⅳ) from the traveler to the bobbin. In this paper we will only consider a more detailed description of region Ⅱ which treats the spinning balloon. Indeed, we find the treatment of essentially all aspects of this problem by Batra, et al. quite satisfactory except for some simplifying assumptions they make regarding the spinning balloon space curve in a quasi-stationary frame of reference rotating with the traveler. The theoretical results of this paper suggest that this part of the problem (region Ⅱ) should be studied more carefully. We believe that the experimental results in the paper by Dr, Peter Lord, which is scheduled after this paper, also support this conclusion. Indeed, these preliminary results suggest that they could form the basis for developing important diagnostic and forecasting tools for ring spinning. In particular, if new tools of this type could be developed, thy would help connect measured fiber calibration properties (and fiber yarn characteristics such as heaviness) and machine setting and operating parameters with yarn quality. These techniques and some related special computer software.

2. Brief Review of Some Results from Kinematics and Vector Analysis

In order to discuss the important physical and geometrical properties of the spinning balloon, we will need to briefly review some results from mechanics and vector analysis as applied to paths of motion and curves in space, Also, this review will facilitate our discussion by introducing the notation and definitions that will be required. In the early part of mechanics courses employing vector methods, many of you were introduced to the following important kinematics result for the acceleration a of a particle along a path of motion (i.e., a trajectory) described by the following parametric equations in Cartesian coordinates $x = x(t)$, $y = y(t)$, and $z = z(t)$ (where r denotes a position vector and t the time):

$$a = \ddot{r} = \ddot{x}i + \ddot{y}j + \ddot{z}k = aT + kv^2N + 0B \qquad (2.1)$$

Here the dots over the symbols denote time differentiation. In (2.1) a is the magnitude of the acceleration, v is the magnitude of the velocity and k is the curvature of this equation, we learn nothing about the motion looking at the components of the acceleration in the given inertial Cartesian coordinate system employing the unit basis vectors (i.j.k). On the other hand, we learn a great deal if we look at the components of frame associated with the trajectory (i.e., the unit tangent vector T = r' = dr/ds, a the unit principal normal vector N(dT/ds = kN), and the unit binormal vector B = T×N, here ds is an infinitesimal interval of are length along the trajectory). Namely, the well-known result that for any particle motion in space at each instant of time, the motion is in the plane of the tangent and principal normal vectors (often referred to as the osculating plane) with the particle undergoing a tangential acceleration a and a normal acceleration kv^2. In particular, we find that the particle is accelerating in the normal direction as if it were instantaneously moving around a circle of radius 1/k tangent to the trajectory at the given point. This circle of "best fit" at any given point in the plane of T and N is called the osculating circle. In general, it follows that the plane of the osculating circle. changes its orientation, and its radius changes as the particle moves along the curve. Roughly speaking, this change in orientation and the corresponding rate of rotation with respect to the arc length of this osculating plane defines the second intrinsic property of a trajectory or a curve in space called the torsion of the curve which will be denoted by the symbol τ. Both k and τ have the dimension of a reciprocal length unit. These and other results that follow are described in more detail in Appendix I of this paper. If we take derivatives with respect to the arc length of the trajectory or space curve, in place of (2.1) we can write the expression:

$$r'' = x''i + y''j + z''k = 0T + kN + 0B \tag{2.2}$$

The differential properties of any particle trajectory or space curve are most fundamentally explained in terms of what are usually called Frenet-Serret Equations:

$$dT/ds = kN \tag{2.3a}$$

$$dN/ds = -(kT+\tau B) \tag{2.3b}$$

$$dB/ds = \tau N \tag{2.3c}$$

In connection with these equations . it should be remarked that in three-dimensional Euclidean space, the curvature k=k(s) and the torsion $\tau = \tau(s)$ provide a complete intrinsic invariant characterization of the properties of any curve. Indeed, they determine the equations of a space curve. Indeed, they determine the equations of a space curve to within translations and rotations of the coordinates employed. Also, for the discussion that follows it is important to keep in mind that the torsion of a curve τ cannot be determined by first and second derivatives of position(e.g., by the velocity and the acceleration). Thus, while the first intrinsic property of a space curve, the curvature k, is determined by second derivatives of the position vector, the second intrinsic property of a curve, the torsion τ, requires third derivatives of the position vector for its determination. In particular, employing (2.3a)-(2.3c) it can be easily seen that the following interesting general result holds:

$$r''' = -k^2T + k'N - k\tau B \tag{2.4}$$

Returning to the description of the spinning balloon curve, we will proceed with the consideration of successive generalizations which provide closer approximations to the observed behavior and which permit us to make use of the intrinsic properties of the spinning balloon curve.

3. The Spinning Balloon Space Curve of Fixed Shape in a Uniformly Rotating Frame of Reference

At the simplest and most idealized level of description of the spinning balloon, we have a quasi-stationary space curve of fixed shape (e.g., given by $\bar{x} = \bar{x}(s), \bar{y} = \bar{y}(s), \bar{z} = \bar{z}(s)$) relative to a uniformly rotating frame of reference denoted by barred coordinates with the z-axis along the spindel axis (taken to coincide with the z-axis of the fixed reference frame of the spinning machine) assumed to rotate with the traveler. Under these special circumstances, we can describe the coordinates of the spinning balloon in the fixed frame of the spinning machine by the equations:

$$\begin{aligned} x(t,s) &= \bar{x}(s)\cos\omega t - \bar{y}(s)\sin\omega t, \\ y(t,s) &= \bar{y}(s)\sin\omega t + \bar{y}(s)\cos\omega t, \\ z(t,s) &= \bar{z}(s) \end{aligned} \qquad (3.1)$$

By contrast, if we were describing the motion of a particle moving in a fixed trajectory $\bar{x} = \bar{x}(t)$, $\bar{y} = \bar{y}(t)$, $\bar{z} = \bar{z}(t)$ with respect to this type of uniformly rotating frame, we would have for the coordinates of the particle in the given nonrotating reference frame :

$$\begin{aligned} x(t,s) &= \bar{x}(s)\cos\theta\omega t - \bar{y}(t)\sin\omega t, \\ y(t,s) &= \bar{y}(t)\sin\omega t + \bar{y}(t)\cos\omega t, \\ z(t,s) &= \bar{z}(s) \end{aligned}$$

If we allow for the possibility that the traveler is not rotating uniformly, them must be replaced by the equations:

$$x(t,s) = \bar{x}(s)\cos\theta(t) - \bar{y}(s)\sin\theta(t),$$

$$y(t,s) = \bar{y}(s)\sin\theta(t) - \bar{z}(s)\cos\theta(t), \qquad (3,2)$$

$$z(t,s) = \bar{z}(s)$$

The typical approach to this aspect of the mechanical problem relating to the mechanics of the balloon is to write out the acceleration terms corresponding to (3.1) or (3.2) and apply Newton's laws of motion to an infinitesimal segment of yarn of length ds whose mass $dm = \lambda ds$ is determined by the linear density of the yarn λ. Neglecting the possibility of the motion of the origin of the rotating frame, one fines the well-known result for the total acceleration in rotating frame :

$$a\ (\text{total}) = \ddot{\bar{r}} + 2\omega \times \dot{\bar{r}} + \alpha \times \bar{r} + \omega \times (\omega \times \bar{r}) \qquad (3.3)$$

where $\omega = \dot{\theta}(t)$ and $\alpha = \ddot{\theta}(t)$. The terms $\dot{\bar{r}}$ and $\ddot{\bar{r}}$ describe the velocity and acceleration of the mass element relative to the rotating frame, and the term involving the expression $2\omega \times \dot{\bar{r}}$ is the Coriolis acceleration which is normally assumed to be negligible. In equation (3.3), the term $\omega \times (\omega \times \bar{r})$ is recognized as the centrifugal acceleration acting in the rotating frame and $\alpha \times \bar{r}$, often called the Euler acceleration, is produced by angular acceleration of the rotating frame. Clearly, the above type of analysis does not bring into evidence the intrinsic properties of the spinning balloon as a space curve type of space curve:

$$\bar{x} = \bar{x}[t,(s)],$$

$$\bar{y} = \bar{y}[t,(s)], \qquad (3.4)$$

$$\bar{z} = \bar{z}[t,(s)]$$

Here (s) represents the functional parameter dependence, which may itself depend on time, which can be eliminated from the equations that describing the spinning balloon curve in the fixed frame of the spinning machine take the form:

$$x(t,s) = \bar{x}[t,(s)]\cos\theta(t) - \bar{y}[t,(s)]\sin\theta(t),$$

$$y(t,s) = \bar{x}[t,(s)]\sin\theta(t) + \bar{y}[t,(s)]\cos\theta(t), \tag{3.5}$$

$$z(t,s) = \bar{z}[t,(s)].$$

In this case one would not expect to be able to neglect the terms $\ddot{\bar{r}}$ and $\dot{\bar{r}}$ in equation (3.3). These considerations have not reached the level of description where we con recognize the explicit contributions to the acceleration terms made by the intrinsic properties of the spinning balloon curve. The next two sections of this paper will be devoted to this task.

4. Intrinsic Description of the Spinning Balloon Curve in the Fixed Frame of the Spinning Machine

Before we actually look at the intrinsic force terms that would be present in the natural local reference provided by the triad of vectors (T,N,B), we want to develop an expression involving the intrinsic accelerations and the rates of change of these intrinsic acceleration terms which will prove to be of much greater value than the expression (3.3) given above.

Employing equation (2.4), reexpressed in terms of time derivatives, the time rate of change of the total acceleration along the (T,N,B) directions described in the fixed frame of reference of the spinning machine is given by:

$$\dot{a}(\text{total}) = (\dot{a} - k^2v^3)T + (\dot{k}v^2 + 3kav)N - k\tau v^3B \tag{4.1}$$

Similarly, for the total time rate of change of the acceleration in the rotating frame of reference, we can write:

$$\dot{a}(\text{total}) = (\dot{\bar{a}} - \bar{k}^2\bar{v}^3)T + (\dot{\bar{k}}\bar{v}^2 + 3\bar{k}\bar{a}\bar{v})N - \bar{k}\bar{\gamma}\bar{v}^3 B + 2\omega \times (\bar{a}T + \bar{k}\bar{v}^2 N)$$

$$+ 3\alpha \times \bar{r} + \dot{\alpha} \times \bar{r} + \alpha \times (\omega \times \bar{r}) + \omega \times (\alpha \times \bar{r}) + \omega \times (\omega \times \dot{\bar{r}}). \qquad (4.2)$$

The external forces acting on a mass element m of the yarn in the spinning balloon are the tension S, the air drag A, and the weight W. These force, together with the boundary conditions and the linear density of the spinning balloon curve in space. In view of the above considerations, it follows that these forces and their time rates of change can best be characterized in terms of the observed intrinsic properties of the spinning balloon curve. The dynamical equations for the mass element m are determined by Newton's second law in the fixed approximately inertial frame of the spinning machine. In an accelerating (non-inertial) frame, the appropriate frame-dependent forces must be used to modify Newton's second law. The external force components and their time rates of change take the form:

$$F_{ext}(\text{total}) = (S_x + A_x)i + (S_y + A_y)j + (S_z + A_z + W_z)k \qquad (4.3a)$$

$$= (\bar{S}_T + \bar{A}_T + \bar{W}_T)T + (\bar{S}_N + \bar{A}_N + \bar{W}_N)N = \bar{F}_T T + \bar{F}_N N \qquad (4.3b)$$

and

$$\dot{F}_{ext}(\text{total}) = (\dot{\bar{F}}_T - \bar{F}_{N\omega k})T + (\dot{\bar{F}}_N + \bar{F}_{T\omega k})N - \bar{F}_{N\omega\tau} B \qquad (4.4)$$

5. The Force Terms Present in General in the Intrinsic Frame of Reference of the Spinning Balloon Curve

Finally, in this section we wish to consider the general frame-dependent force terms present in the local intrinsic frame of reference (T,N,B) acting on an infinitesimal mass element m of the spinning balloon. In follows that these general frame-dependent force terms are produced as a result of the corresponding external force terms acting on the mass elements forming the spinning balloon curve. These terms can be found by construction the appropriate transformations to the intrinsic frame of reference and then formulating the kinetic energy of the balloon mass element in the intrinsic frame and substituting the result into the general form of Lagrange's equations of motion. The detailed derivation of these results are given in Appendix II of this paper. We fine for the components of these results, expressed in local Cartesian coordinates (x,y,z) corresponding to the basis vectors (T,N,B), the expressions (where $\omega_k = kv$ and $\omega_\tau = \tau v$) :

$$m(\ddot{\bar{x}} + a - 2\omega_k \dot{\bar{y}} - \dot{\omega}_k \bar{y} - \omega_\tau^2 \bar{z} - \omega_k \omega_\tau \bar{z} = \bar{F}_x , \tag{5.1}$$

$$m(\ddot{\bar{y}} + 2\omega_k \dot{\bar{x}} + 2\omega_\tau \dot{\bar{z}} + \dot{\omega}_k \bar{x} + \dot{\omega}_\tau \bar{z} - \omega_k^2 \bar{y} - \omega_\tau^2 \bar{y} + \omega_k v) = \bar{F}_y , \tag{5.2}$$

$$m(\ddot{\bar{z}} - 2\omega_\tau \dot{\bar{y}} - \dot{\omega}_\tau \bar{y} - \omega_\tau^2 \bar{z} - \omega_k \omega_\tau \bar{x}) = \bar{F}_z \ . \tag{5.3}$$

As would be expected, for a mass particle on the spinning balloon curve (at the origin of the coordinate system) these results simply reduce to the result (2.1) give at the beginning of this paper. For a mass particle a small distance away from this origin (even if it is at rest in the intrinsic frame), we see that a number of frame-dependent force terms are present which relate to the intrinsic properties k and τ of the spinning balloon curve. Certainly, these types of frame-dependent force terms would be acting on the fibers or fiber clumps located off the axis of the fiber yarn. Clearly, the effects of such terms and the corresponding external forces and boundary conditions must account for the behavior of the spinning balloon curve. If one considers the intrinsic angular velocities ω_k and ω_τ above in terms of the familiar results obtained for a pure time dependent rotation, it is seen by comparison that the familiar centrifugal, Coriolis, and Euler forces are similar in form to many of the terms of he equations of motion (5.1)-(5.2). Hence, the terms in (5.1)-(5.3) which are linear in the first time derivatives of the coordinates may be called "Coriolis -like", while the terms linear in the coordinates are either "Euler-lime" or "centrifugal -like".

6. Summary and Conclusions

In future work we hope to apply these results to better design experiments of studying the spinning balloon time-changing curve in terms of its intrinsic properties. Also, we expect to be able to apply these results to better analyze the dynamics of the spinning balloon with the help of the computer studies employing Mathematical. Finally, in collaboration with the experimental studies and developments of our research group, we hope to develop some practical diagnostic and forecasting tools for the spinning process.

APPENDIX Ⅰ

SOME PROPERTIES OF SPACE CURVES

Any curve in three-dimensional Euclidean space can be represented as the locus of the end points of a vector r = r(s) from the origin of a Cartesian coordinate system where r is expressed parametrically in terms of the arc length s of the curve:

$$r(s)=x(s)i + y(s)j + z(s)k. \tag{AI.1}$$

Here i, j and k are the unit vectors along the x, y and z axes of the Cartesian coordinate system. It is well-known that the square of an infinitesimal internal of arc length along the curve is given by the expression

$$ds^2 = dx^2 + dy^2 + dz^2 \tag{AI.2}$$

In the case of the motion of a particle moving along a trajectory generating a space curve, time t may be taken as the parameter along the curve where the magnitude of the velocity is given by $v = dx/dt$.

The unit tangent vector to the curve is T = dr/ds =v/v. Given T.T=1, it is easily shown that the vector dT/ds is perpendicular to the tangent vector:

$$\frac{d}{ds}(T \cdot T) = T \cdot \frac{dT}{ds} = 0.$$

The principal normal N to the curve is a unit vector in the direction of dT/ds and is defined as

$$N=(1/k)(dT/ds). \tag{AI.2}$$

Here k is the curvature, and its reciprocal ρ is the radius of curvature. The curvature of a curve can be roughly defined as the rate of change of the direction of a curve with respect to its length. More precisely, it can be described in terms of the angle $\Delta\theta_T$ between two nearby tangent vectors to the curve at two nearby points on the curve separated by Δs, where

$$\text{Curvature} = k = \lim_{\Delta s \to 0}\left(\frac{\Delta\theta_T}{\Delta s}\right) = \frac{d\theta_T}{ds}$$

Clearly, k has the unit of reciprocal length. In accord with (AI.3) and the above definition the curvature can be easily found in terms of the following expression:

$$k = \left[\left(\frac{d^2x}{ds^2}\right)^2 + \left(\frac{d^2y}{ds^2}\right)^2 + \left(\frac{d^2z}{ds^2}\right)^2\right]^{1/2} \tag{AI.4}$$

The binormal vector B, which is unit vector perpendicular to both T and N, is defined by B=T×N. Thus, at each point along the curve r(s) the three vectors T, N, and B from a well-defined orthogonal triad. It can be shown that the rates of change of the principal normal vector N and the binormal vector B with respect to s are given by the expressions $dN/ds = -(_{\tau}B +_{k} T)$ and $dB/ds = \tau N$, where τ is called the torsion of the space curve. The torsion of a curve can be roughly defined as the rate of change of the position of the osculating plane, which is defined by tangent and normal vectors to the curve, with respect to the curve. More precisely, it can be described in terms of the angle $\Delta\theta_B$ between two nearby binormal vectors to the curve at two points on the curve separated by Δs, where

$$\text{torsion} = \tau = \lim_{\Delta s \to 0}\left(\frac{\Delta\theta_B}{\Delta s}\right) = \frac{d\theta_B}{ds}$$

Like the curvature, torsion has the unit reciprocal length. Thus, $1/\tau$ is often referred to as the radius of the torsion τ. The above-mentioned expressions for dT/ds, dN/ds, and which we bring together here are often referred to as the Frenet-Serret formulas:

$$\frac{dT}{ds} = kT, \tag{AI.5}$$

$$\frac{dN}{ds} = -(_{k}T +_{\tau} N), \tag{AI.6}$$

$$\frac{dB}{ds} = \tau N. \tag{AI.7}$$

These remarkable relations are of fundamental significance to differential geometry and to kinematics in that they characterize differentially the basic geometric properties of paths of motion. Indeed, it is well-known that the curvature k=k(s) and torsion $\tau = \tau(s)$ determine the intrinsic invariant properties of all space curves given the background geometry. In fact, given k =k(s) and $\tau = \tau(s)$, the equations of any space curve in three dimensional Euclidean space are determined to within a translation and a rotation of the coordinate system being employed to describe the space curve.

The triad of unit vectors T,N,B which is well defined at each point of a space curve, is of considerable utility in studying certain properties of the paths of motion of particles or rigid bodies. Indeed, it serves to define a natural local coordinate system at each point along the path. Of course, in the case of particle motion, the T,N,B vector triad forms a local generally non inertial reference frame because it undergoes accelerated motion as it moves along the curve following the motion of the given particle. This generally non inertial local reference system will be called the intrinsic frame of reference of the path of motion since it follows that the intrinsic properties of a space curve (or path of motion) are most simply defined in terms of this frame. Furthermore, the Frenet-Serret formulas referred to this frame of this frame. Furthermore, the Frenet-Serret formulas referred to this frame of reference are very useful in formulating the characteristic forces (i.e., the frame-dependent forces) arising in this frame because of the accelerations it experiences.

APPENDIX Ⅱ

THE EQUATIONS OF MOTION O A POINT MASS FORMULATED WITH RESPECT TO THE INTRINSIC FRAME OF REFERENCE

This appendix will be devoted exclusively to obtaining the force terms occurring in the equations of motion of a particle. These force terms are formulated in the local intrinsic frame of reference $\overline{\mathrm{K}}$ which is defined by a general path described with respect to some given inertial frame $\boldsymbol{K}$. First a transformation if made to a reference frame $\widetilde{\mathrm{K}}$ such that the axes of $\widetilde{\mathrm{K}}$ remain parallel to their counterparts in the inertial frame K. The origin of $\widetilde{\mathrm{K}}$ coincides at all times t with the origin of the intrinsic frame $\widetilde{\mathrm{K}}$ of the given path of motion determined by the functions $\mathrm{x}_0(\mathrm{s}), \mathrm{y}_0(\mathrm{s}), \mathrm{Z}_0(\mathrm{s})$ where the parameter s is the element of arc along the path. The unbarred coordinates of the $\boldsymbol{K}$ frame are thus related to the coordinates of $\widetilde{\mathrm{K}}$ by the expressions.

$$\mathrm{x} = \mathrm{x}_0(\mathrm{s}) + \widetilde{\mathrm{x}}\,,$$

$$y = y_0(s) + \tilde{y}, \qquad \text{(A II.2)}$$

$$z = z_0(s) + \tilde{z}.$$

It can be seen that the components of the vectors T,N, and B provide the nine direction cosines specifying the time dependent rotation from the $\bar{\boldsymbol{K}}$ to the successive $\boldsymbol{K}$ frame :

$$\tilde{x} = T_x\bar{x} + N_x\bar{y} + B_x\bar{z},$$

$$\tilde{y} = T_y\bar{x} + N_y\bar{y} + B_y\bar{z}, \qquad \text{(A II.2)}$$

$$\tilde{z} = T_z\bar{x} + N_z\bar{y} + B_z\bar{z}$$

In order to derive the equations of motion of the particle in terms of the coordinates of the intrinsic frame $\overline{K}$, the kinetic energy must by expressed as a function of the barred coordinates. In the inertial system $\boldsymbol{K}$, the kinetic energy has the form

$$T = \frac{1}{2}m(ds/dt)^2 = \frac{1}{2}m\left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2\right] \qquad \text{(A II.3)}$$

The kinetic energy may be expressed in the $\overline{K}$ coordinates of the intrinsic frame by using the transformations (A II.1) and (A II.2). It should be noted that the coordinate differentials as formed below introduce the magnitude of the instantaneous velocity $v = ds/dt$ of the origin of the intrinsic frame $\overline{K}$ with respect to the inertial frame $\boldsymbol{K}$. For the motion of a particle, the unit vectors T, N, B of the intrinsic frame $\overline{K}$ change orientation from point to point along the trajectory as functions of the time. Employing the Frenet-Serret formulas introduced in Appendix Ⅰ, one can write

$$dx = T_x vdt + kN_x v\bar{x}dt + T_x d\bar{x} - (\tau B_x + kT_x)v\bar{y}dt + N_x d\bar{y} + \tau N_x v\bar{z}dt + B_x d\bar{z}, \qquad \text{(A II.4)}$$

$$dy = T_y vdt + kN_y v\bar{x}dt + T_y d\bar{x} - (\tau B_y + kT_y)v\bar{y}dt + N_y d\bar{y} + \tau N_y v\bar{z}dt + B_y d\bar{z}, \qquad \text{(A II.5)}$$

$$dz = T_z vdt + kN_z v\bar{x}dt + T_z d\bar{x} - (\tau B_z + kT_z)v\bar{y}dt + N_z d\bar{y} + \tau N_z v\bar{z}dt + B_z d\bar{z}. \qquad \text{(A II.6)}$$

The kinetic energy of the particle in terms of the coordinates of the intrinsic $\bar{K}$ frame is found by substituting the coordinate differentials (AⅡ.4), (AⅡ.5),(AⅡ.6) into (AⅡ.3). The simplification of the resulting expression is facilitated by the relations which can be obtained as a consequence of the mutual orthogonality of the unit vectors T, N, and B. The expression for the kinetic energy is given by:

$$T = \frac{1}{2} m\left[\dot{\bar{x}}^2 + \dot{\bar{y}}^2 + \dot{\bar{z}}^2 + 2\omega_k (\bar{x}\dot{\bar{y}} - \bar{y}\dot{\bar{x}}) - 2\omega_\tau (\bar{y}\dot{\bar{z}} - \bar{z}\dot{\bar{y}})\right]$$

$$+ 2\dot{\bar{x}}v + \omega_k^2 (\bar{x}^2 + \bar{y}^2) + \omega_\tau^2 (\bar{y}^2 + \bar{z}^2) + 2\omega_k \omega_\tau \bar{x}\bar{z} - 2\omega_k v\bar{y} + v^2, \quad (\text{A}Ⅱ.7)$$

Where $\omega_k = kv$ and $\omega_\tau = \tau v$

The equation of motion of the mass particle expressed in terms of the intrinsic $\bar{K}$ frame may now be found using Lagrange's equations in the form:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\bar{x}}}\right) - \frac{\partial T}{\partial \bar{x}} = \bar{Q}_x,$$
$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\bar{y}}}\right) - \frac{\partial T}{\partial \bar{y}} = \bar{Q}_y, \quad (\text{A}Ⅱ.8)$$
$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\bar{z}}}\right) - \frac{\partial T}{\partial \bar{z}} = \bar{Q}_z$$

where the generalized forces $\bar{Q}_x, \bar{Q}_y$ and $\bar{Q}_z$ are expressed in terms of the coordinates of the $\bar{K}$ frame. From (AⅡ.7) and (AⅡ.8) the equations of motion are :

$$m(\ddot{\bar{x}} + a - 2\omega_k \ddot{\bar{y}} - \omega_k \bar{y} - \omega_k^2 \bar{x} - \omega_k \omega_\tau \bar{z}) = \bar{Q}_x, (\text{A}Ⅱ.9)$$

$$m(\ddot{\bar{y}} + 2\omega_k \dot{\bar{x}} + 2\omega_\tau \dot{\bar{z}} + \dot{\omega}_k \bar{x} + \dot{\omega}_\tau \bar{z} - \omega_k^2 \bar{y} - \omega_\tau^2 \bar{y} + \omega_k v) = \bar{Q}_y, (\text{A}Ⅱ.10)$$

$$m(\ddot{\bar{z}} - 2\omega_\tau \dot{\bar{y}} - \dot{\omega}_\tau \bar{y} - \omega_\tau^2 \bar{z} - \omega_k \omega_\tau \bar{x} = \bar{Q}_z, (\text{A}Ⅱ\ 11)$$

where $a = d^2s/dt^2$

If one considers the effective intrinsic angular velocities ω_κ and ω_τ above in terms of the familiar results obtained for a pure time dependent rotation, it is seen by comparison that the familiar centrifugal, Coriolis, and Euler forces are similar in form to many of the terms of the equations of motion (AⅡ.9)-(AⅡ.11). Hence, the terms in (AⅡ.9)-(AⅡ.11), which are linear in the first time derivatives of the coordinates, may be called "Coriolis-like", while the terms linear in the coordinates are either "Euler-like" or "centrifugal-like".